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Note

A Noncompact Topological Minimax Theorem

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Using the minimax inequality of Fan [F], Ha [H2] obtained a noncompact version of Sion's minimax theorem [S] in topological vector spaces. In this note, we extend Ha's theorem to a topological minimax theorem without requiring any linear structure on the underlying spaces. We assume all topological spaces are Hausdorff.

Komiya [K] proved a topological minimax theorem that also generalizes Sion's minimax theorem and another minimax theorem of Ha [H1]. A two-function version of Komiya's theorem is given in [LQ]. The following result is a consequence of the theorem in [LQ].

THEOREM 1. *Let X and Y be topological spaces and let $f, g: X \times Y \to \mathbb{R}$. Assume that*

(i) *$f(x, y) \leqslant g(x, y)$ for all $x \in X$, $y \in Y$;*

(ii) (a) *for all x in X, $f(x, \cdot)$ and $g(x, \cdot)$ are lower semi-continuous on Y;*
(b) *for all y in Y, $f(\cdot, y)$ and $g(\cdot, y)$ are upper semi-continuous on X;*

(iii) (a) *for all $y_1, ..., y_n$ in Y and for all $\beta_1, ..., \beta_n$ in $\mathbb{R}$, the set $\{x : x \in X, f(x, y_i) \geqslant \beta_i, i = 1, ..., n\}$ is either connected or empty;*
(b) *for all y_1, y_2 in Y, there exists a connected set C containing y_1, y_2 and for all y in C and all x in X,*

$$g(x, y) \leqslant f(x, y_1) \vee g(x, y_2);$$

(iv) *X is compact; then*

$$\sup_X \inf_Y g(x, y) \geqslant \inf_Y \sup_X f(x, y).$$

We give a noncompact version of Theorem 1 by following Ha's idea.

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THEOREM 2. *Let X and Y be topological spaces and let $f, g: X \times Y \to \mathbb{R}$ satisfy conditions* (i), (ii), (iii) *in Theorem* 1 *and*

(iv′) *There exists a subset K in X and a nonempty compact subset H in Y such that*

$$\inf_{Y \setminus H} \sup_{K} g(x, y) \geqslant \inf_{Y} \sup_{X} f(x, y) \tag{$*$}$$

and for any finite subset F in X, there is a compact set $K(F)$ containing F and K with the property that for any $y_1, ..., y_n$ in Y and for any $\beta_1, ..., \beta_n$ in $\mathbb{R}$, the set $\{x : x \in K(F),\ f(x, y_i) \geqslant \beta_i,\ i = 1, ..., n\}$ is either connected or empty.

Then

$$\sup_{X} \inf_{Y} g(x, y) \geqslant \inf_{Y} \sup_{X} f(x, y).$$

Proof. Let $x = \sup_X \inf_Y g(x, y)$ and $\beta = \inf_Y \sup_X f(x, y)$. Suppose $\beta > \alpha$. Choose $\varepsilon > 0$ such that $\beta > \alpha + \varepsilon$. For $x \in X$, $h = f, g$, let $L_h(x) = \{y : y \in Y, h(x, y) \leqslant \alpha + \varepsilon\}$ and $L(x) = L_f(x) \cap (\bigcap_{k \in F} \mathrm{Lg}(k))$. Then $\{L(x)\}_{x \in X}$ is a family of closed subsets in Y. We claim that for any $x_1, ..., x_n$ in X, $\bigcap_{i=1}^{n} L(x_i) \neq \varnothing$.

Let $F = \{x_1, ..., x_n\}$ be a finite subset in X. Let $K(f)$ be a compact subset in X that satisfies the properties in (iv′). By Theorem 1, we have

$$\sup_{K(F)} \inf_{Y} g(x, y) \geqslant \inf_{Y} \sup_{K(F)} f(x, y).$$

Hence $\alpha + \varepsilon > \alpha \geqslant \sup_{K(F)} \inf_Y g(x, y) \geqslant \inf_Y \sup_{K(F)} f(x, y)$. Thus $\bigcap_{i=1}^{n} L(x_i) \supset \bigcap_{x \in K(F)} L(x) = \varnothing$.

Now, $\inf_{Y \setminus H} \sup_K g(x, y) \geqslant \inf_Y \sup_X f(x, y) = \beta > \alpha + \varepsilon$, hence $\bigcap_{n \in K} \mathrm{Lg}(k) \subset H$. Since H is compact, it follows that $\bigcap_X L(x) \neq \varnothing$. But this implies that $\inf_Y \sup_X f(x, y) \leqslant \alpha + \varepsilon < \beta$. This contradicts the definition of β.

COROLLARY 1. *Let X and Y be topological spaces and let $f: X \times Y \to \mathbb{R}$. Assume that*

(1) *for all x in X, $f(x, \cdot)$ is lower semi-continuous on Y and for all y in Y, $f(\cdot, y)$ is upper semi-continuous on X;*

(2) (a) *for all $y_1, ..., y_n$ in Y and for all $\beta_1, ..., \beta_n$ in $\mathbb{R}$, the set $\{x : x \in X, f(x, y_i) \geqslant \beta_i, i = 1, ..., n\}$ is either connected or empty;*

(b) *for all y_1, y_2 in Y, there exists a connected set C containing y_1, y_2 and for all y in C and for all x in X,*

$$f(x, y) \leqslant f(x, y_1) \vee f(x, y_2);$$

(3) *There exists a subset K in X and a compact set H in Y such that*

$$\inf_{Y\setminus H} \sup_{K} f(x, y) \geqslant \inf_{Y} \sup_{X} f(x, y)$$

and for any finite subset F in X, there is a compact set K(F) containing K and F with the property that for any $y_1, ..., y_n$ in Y and for any $\beta_1, ..., \beta_n$ in $\mathbb{R}$, the set $\{x : x \in K(F),\ f(x, y_i) \geqslant \beta_i,\ i = 1, ..., n\}$ is either connected or empty. Then

$$\sup_{X} \inf_{Y} f(x, y) = \inf_{Y} \sup_{X} f(x, y).$$

Proof. We only need to observe that in the case $H = \varnothing$. Then

$$\inf_{Y} \sup_{X} f(x, y) \leqslant \inf_{Y\setminus H} \sup_{K} f(x, y) = \inf_{Y} \sup_{K} f(x, y) \leqslant \inf_{Y} \sup_{K(F)} f(x, y)$$
$$\leqslant \inf_{Y} \sup_{X} f(x, y).$$

Hence, in the proof of Theorem 2, after applying Theorem 1, we have

$$\inf_{Y} \sup_{X} f(x, y) = \inf_{Y} \sup_{K(F)} f(x, y) \leqslant \sup_{K(F)} \inf_{Y} f(x, y) \leqslant \sup_{X} \inf_{Y} f(x, y).$$

It follows that $\inf_Y \sup_X f(x, y)$ and the proof is complete. ∎

COROLLARY 2 (Ha). *Let X and Y be convex sets in topological vector spaces and let $f: X \times Y \to \mathbb{R}$. Assume that,*

(i) *for all x in X, $f(x, \cdot)$ is lower semi-continuous and quasi-convex on Y;*

(ii) *for all y in Y, $f(\cdot, y)$ is upper semi-continuous and quasi-concave on X;*

(iii) *there exist a nonempty compact convex set K in X and a compact set H in Y such that*

$$\inf_{Y\setminus H} \sup_{K} f(x, y) \geqslant \inf_{Y} \sup_{X} f(x, y).$$

Then

$$\sup_{X} \inf_{Y} f(x, y) = \inf_{Y} \sup_{X} f(x, y).$$

Proof. To check that the conditions in Corollary 1 are satisfied.

(1) Is clear.

(2a) Follows from the quasi-concavity of $f(\cdot, y)$.

(2b) Follows from the quasi-convexity of $f(x, \cdot)$. Let y_1 and y_2 be fixed elements in Y. For each x in X, let $C=\bigcap_X \{y : y \in Y, f(x, y) \leqslant f(x_1, y_1) \vee f(x, y_2)\}$. Then C is a convex set containing y_1, y_2 and clearly satisfies the conditions in (2b).

(3) For any finite set F in X, let $K(F)$ be the convex hull of K and F. Then $K(F)$ is a compact set satisfying the conditions in (3).

Remarks. (1) If the inequality $(*)$ in (iv′) is replaced by the stronger condition,

$$\inf_{Y \setminus H} \sup_{K} f(x, y) \geqslant \inf_{Y} \sup_{X} f(x, y),$$

then even without requiring that $g(x, \cdot)$ be lower semi-continuous on Y, the conclusion of Theorem 2 still holds. In the proof of Theorem 2, we need only replace $L(x)$ by the set $L_f(x) \cap (\bigcap_{k \in K} L_f(k))$ for all x in X.

(2) In the case that X is compact, it is interesting to compare Corollary 1 with the minimax theorem of Wu [W]. Wu's theorems were simplified, independently by Tuy [T] and by Geraghty and Lin [GL]. In [GL], by requiring a connectedness condition stronger than that in (2b), then without requiring that $f(x, \cdot)$ be lower semi-continuous on Y for all x in X, the minimax theorem also holds.

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